

# A PLANE-SYMMETRIC GRAVITATIONAL FIELD WITH MATTER AS A GENERALIZED NONLINEAR SIGMA MODEL

S.V. Chervon<sup>1</sup> and D.Yu. Shabalkin<sup>2</sup>

Ulyanovsk State University, 42 Leo Tolstoy St., Ulyanovsk 432700, Russia

The Einstein equations for a plane-symmetric gravitational field coupled to an arbitrary nonlinear sigma model (NSM) are shown to be represented in the form of dynamical equations of a *generalized effective NSM*. The gravitational equations are studied in this case by the methods of analyzing NSM equations. In the case of a two-component diagonal NSM exact solutions are found by the functional parameter method.

PACS: 04.20.-q, 04.20.Jb

## 1 Introduction

It was proposed in Ref. [2] (see also [3]) to consider the plane-symmetric gravitational field as a four-component nonlinear sigma model (NSM) defined on a two-dimensional space-time. It was also shown that some progress can be achieved using the sigma-model approach. Nevertheless, it was difficult to obtain a suitable form of presentation for solutions: the final result of [2] involves integration over an elliptic function. To avoid this problem, the functional parameter method recently developed [4] and successfully applied to an effective NSM for plane-symmetric and axially symmetric gravitational fields [6]. The problem of strong correspondence between the Einstein equations and the dynamical equations of a four-component NSM was analyzed in detail in a PhD thesis of one of the authors [7]. Taking into account these results, we intend in the present paper to generalize the NSM method described above to the case of a plane-symmetric space-time filled by chiral fields. The two-component NSM as a source of gravity is discussed in detail.

### 1.1 Effective NSM for a vacuum plane-symmetric gravitational field

It was shown in [2] that the NSM method may be applied for studying the Einstein equations for a vacuum plane-symmetric gravitational field. This connection was introduced with the following construction (see for details [3]).

The metric

$$ds^2 = Adx^2 + 2Bdx dy + Cdy^2 - D[dz^2 - dt^2] \quad (1)$$

was chosen. The metric coefficients  $A$ ,  $B$ ,  $C$ ,  $D$  depend on  $z$  and  $t$  only. Following the ideas of the effective NSM method, let us introduce an NSM with the chiral fields  $\varphi^1 = \psi$ ,  $\varphi^2 = \theta$ ,  $\varphi^3 = \chi$ ,  $\varphi^4 = \phi$ , defined

on the space-time  $g^{(e)}: ds_{(e)}^2 = dz^2 - dt^2$ , and taking their values in the target space

$$h_{AB}^{(e)} = e^\psi \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sinh^2 \theta & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (2)$$

The dynamic equations derived from the action

$$\mathcal{S} = \int_{\mathcal{M}} \sqrt{|g^{(e)}|} d^4x \left\{ \frac{1}{2} h_{IK}^{(e)}(\varphi) \varphi_{,i}^I \varphi_{,k}^K g^{(e)ik} \right\} \quad (3)$$

have the form

$$\begin{aligned} \square e^\psi &= 0, \\ \square \theta + (\psi_z \theta_z - \psi_t \theta_t) - \frac{1}{2}(\chi_z^2 - \chi_t^2) \sinh 2\theta &= 0, \\ \square \chi + 2(\theta_z \chi_z - \theta_t \chi_t) \coth \theta + (\psi_z \chi_z - \psi_t \chi_t) &= 0, \\ \square \phi + \frac{1}{2}(\psi_z^2 - \psi_t^2) - \frac{1}{2}(\chi_z^2 - \chi_t^2) \sinh^2 \theta \\ &\quad - \frac{1}{2}(\theta_z^2 - \theta_t^2) = 0, \\ \square &\equiv \partial_{zz} - \partial_{tt}. \end{aligned} \quad (4)$$

The Einstein vacuum equations  $R_{ik} = 0$  for the metric (1) are reproduced from the dynamic equations (4) if the metric coefficients are connected with the chiral fields in the following way:

$$\begin{aligned} A &= -e^\psi (\cos \chi \sinh \theta + \cosh \theta), \\ B &= e^\psi \sin \chi \sinh \theta, \\ C &= e^\psi (\cos \chi \sinh \theta - \cosh \theta), \\ D &= e^\phi. \end{aligned} \quad (5)$$

This direct connection between the gravitational field and the effective chiral fields allows one to transfer the analysis from the Einstein vacuum equations for (1) to the effective NSM equations (4).

The effective NSM equations (4) were studied by means of the isometric ansatz method and the functional parameter method [2, 5, 4, 6]. The functional parameter method provides a wider family of solutions and should be presented here for further constructions. Solutions of (4) will be sought in the form

$$\theta = \theta(\xi), \quad \chi = \chi(\xi), \quad \phi = \phi(\xi), \quad \psi = \ln \xi \quad (6)$$

<sup>1</sup>e-mail: chervon@sv.uven.ru

<sup>2</sup>e-mail: sd@sv.uven.ru

where  $\xi = \xi(z, t)$  is a functional parameter satisfying

$$\square \xi = 0, \quad (7)$$

according to the first equation (4).

Substitution of (6) into the other equations (4) leads to the set of ordinary differential equations

$$\begin{aligned} \ddot{\theta} + \frac{1}{\xi} \dot{\theta} - \frac{1}{2} \dot{\chi}^2 \sinh 2\theta &= 0, \\ \ddot{\chi} + \frac{1}{\xi} \dot{\chi} + 2\dot{\chi} \dot{\theta} \coth \theta &= 0, \\ \ddot{\phi} + \frac{1}{2\xi^2} - \frac{1}{2} \dot{\theta}^2 - \frac{1}{4} \dot{\chi}^2 \sinh^2 \theta &= 0 \end{aligned} \quad (8)$$

for the functions  $\theta$ ,  $\chi$  and  $\phi$ . The parameter  $\xi$  is treated here as an independent variable. Here and henceforth

$$\dot{\phi}_a = \frac{d}{d\xi} \phi_a, \quad \ddot{\phi}_a = \frac{d^2}{d\xi^2} \phi_a, \dots; \quad a = 1, 2, 3, 4.$$

Eqs. (8) are exactly integrated and the solutions may be written as

$$\psi = \ln \xi, \quad \square \xi(z, t) = 0; \quad (9)$$

$$\theta = \text{Arcosh} \left[ \frac{k}{2} \left( [\xi/\xi_0]^a + [\xi/\xi_0]^{-a} \right) \right]; \quad (10)$$

$$\chi_{\pm} = \arctan \chi_0 \pm \arctan \left[ \left| \frac{a}{c} \right| \frac{[\xi/\xi_0]^{2a} + 1}{[\xi/\xi_0]^{2a} - 1} \right]; \quad (11)$$

$$\phi = \phi_0 + \phi_1 \xi + \frac{a^2 - 1}{2} \ln \xi, \quad (12)$$

where  $k = \sqrt{(a^2 + c^2)/a^2}$ ;  $a$  and  $c$  are arbitrary constants. The metric is obtained by substituting (9)–(12) into (5).

The case of vacuum axially and plane-symmetric space-times can be thus investigated by means of the functional parameter method [6].

An evident generalization of the effective NSM method is to plane-symmetric gravitational field filled by matter represented by physical fields, e.g., chiral ones. It is preferred to study such systems by NSM analysis only. In other words, construction of a generalized effective NSM, including gravity as well as a physical NSM, is desired. In Sec. 2 we consider the Einstein equations  $R_{ik} = T_{ik} - \frac{1}{2} g_{ik} T$  for the energy momentum tensor  $T_{ik}$  corresponding to an arbitrary  $M$ -component NSM. The possibility of representing both the Einstein equations and the NSM dynamic equations in terms of a *generalized* effective NSM will be analyzed.

A complete analysis of a plane-symmetric gravitational space filled with a 2-component nonlinear sigma model is carried out in Sec. 3. Exact solutions are presented there.

## 2 Generalization of the effective NSM method

Consider a space-time (1) created by an NSM described by the chiral fields  $\Phi^1(z, t), \dots, \Phi^M(z, t)$ , taking their values on some  $h_{IK}(\Phi)$ . Let us accept the following rules for the indices:  $A, B = 1, \dots, 4$  and  $I, J, K = 1, \dots, M$ .

The system under consideration will be governed by the Einstein equations

$$R_{ik} = T_{ik} - \frac{1}{2} g_{ik} T \quad (13)$$

and the NSM field equations

$$\begin{aligned} h_{JK} \square \Phi^K + \frac{1}{\alpha} (\alpha_t \Phi_t - \alpha_z \Phi_z) \\ + (h_{IK, J} - \frac{1}{2} h_{JK, I}) (\Phi_t^I \Phi_t^K - \Phi_z^I \Phi_z^K) = 0, \\ \alpha = \sqrt{AC - B^2}. \end{aligned} \quad (14)$$

To construct a generalized effective NSM it is necessary:

- to represent the gravity equations (13) as dynamic equations of a certain NSM, i.e. to define an effective space-time ( $ds_{(e)}^2$ ) containing effective chiral fields, the effective chiral space  $h_{IK}^{(e)}$  and a connection between the metric coefficients and the chiral fields similar to the vacuum case;
- to add the physical NSM (14) into this model.

Having satisfied these requirements, one can construct a generalized chiral space in the form

$$H_{LN} = H_{LN}(h_{AB}^{(e)}, h_{IK}), \quad (L, N, P = 1 \dots M + 4)$$

with the fields  $\Psi = (\varphi^1, \dots, \varphi^4, \Phi^1, \dots, \Phi^M)$  defined in the space-time with  $g_{ik}^{(e)}$  and the corresponding Lagrangian

$$\mathcal{L} = \frac{1}{2} \sqrt{|g^{(e)}|} g^{(e) ik} H_{LN} \Psi_i^L \Psi_k^N. \quad (15)$$

The equations due to (15)

$$\frac{1}{\sqrt{|g^{(e)}|}} (H_{NL} \Psi_k^L g^{(e) ik})_i - \frac{1}{2} H_{NL, P} \Phi_i^L \Phi_k^P g^{(e) ik} = 0 \quad (16)$$

should unify (13) and (14).

Let us show now that the Einstein equations are contained in the chiral model equations. Eqs. (13) with the chosen energy-momentum tensor may be written as  $R_{ik} = h_{IK} \Phi_i^I \Phi_k^K$ . For  $i, k = 1, 2$  they are the same as in the vacuum case  $R_{ik} = 0$  and therefore may be represented as (16). They follow from (16) with

$N = 2, 3, 4$ . The equation with  $N = 1$  will be satisfied by the Einstein equations with  $i, k = 3, 4$ .

The original set of equations (13), (14) in terms of the NSM will be written in the following way:

$$\begin{aligned} \square e^\psi &= 0, \\ \square \theta + (\psi_z \theta_z - \psi_t \theta_t) - \frac{1}{2}(\chi_z^2 - \chi_t^2) \sinh 2\theta &= 0, \\ \square \chi + 2(\theta_z \chi_z - \theta_t \chi_t) \coth \theta + (\psi_z \chi_z - \psi_t \chi_t) &= 0, \\ \square \phi + \frac{1}{2}(\psi_z^2 - \psi_t^2) - \frac{1}{2}(\chi_z^2 - \chi_t^2) \sinh^2 \theta \\ - \frac{1}{2}(\theta_z^2 - \theta_t^2) - h_{AB}(\Phi_t^A \Phi_t^B - \Phi_z^A \Phi_z^B) &= 0, \end{aligned} \quad (17)$$

$$\begin{aligned} h_{JK} \square \Phi^K + (\psi_t \Phi_t - \psi_z \Phi_z) \\ + (h_{IK,J} - \frac{1}{2} h_{JK,I})(\Phi_t^I \Phi_t^K - \Phi_z^I \Phi_z^K) &= 0, \\ J = 1 \dots M. \end{aligned} \quad (18)$$

All the equations are NSM dynamic equations. It is now necessary to unify them in a generalized effective model corresponding to (15) by choosing the form of  $H_{LN}$  and  $g_{ik}^{(e)}$ . Eqs.(14) give the following possible form of the generalized chiral space metric:

$$H_{LN} = \begin{pmatrix} h_{AB}^{(e)} & 0 \\ 0 & H_{IK} \end{pmatrix}. \quad (19)$$

Eq. (17.4), containing the term  $h_{IK}(\Phi_t^I \Phi_t^K - \Phi_z^I \Phi_z^K)$ , can arise from the generalized effective NSM with the chiral space metric (19) if

$$H_{IK} = -2e^\psi h_{IK}. \quad (20)$$

Note that Eqs. (17) are constructed under the assumption  $g^{(e)} = \text{diag}(1, -1)$  but with the metric  $g_{ik}$  for the physical NSM [Eqs. (18) have the form (1)]. However, the NSM equations with the physical chiral fields  $\Phi^I$ , defined on  $g^{(e)} = \text{diag}(1, -1)$  with the chiral space in the form (20), will coincide with the physical NSM equations (14).

In this way the NSM formed by the chiral fields

$$\Psi^L = (\varphi^1, \dots, \varphi^4, \Phi^1, \dots, \Phi^M), \quad (21)$$

defined on the space-time with the metric

$$ds_{(e)}^2 = dz^2 - dt^2 \quad (22)$$

with the chiral space corresponding to

$$\begin{aligned} dS^2 &= e^\psi (-d\psi^2 + d\theta^2 + \sinh^2 \theta d\chi^2 \\ &\quad - 2d\psi d\phi - 2h_{IK} d\Phi^I d\Phi^K) \end{aligned} \quad (23)$$

may be treated as a generalized effective nonlinear sigma model of the plane-symmetric space-time (1) filled with a physical NSM of general form. The dynamic equations (17), (18) of the generalized effective NSM (21-23) contain the Einstein equation (13) for the metric (1) with the physical NSM (13) and the dynamic equations of physical chiral fields (14).

### 3 Exact solutions

Consider a special case of a physical NSM, namely, a two-component NSM, represented by  $\Phi^1 = \sigma(z, t)$ ,  $\Phi^2 = \rho(z, t)$  and

$$h_{AB} = \begin{pmatrix} P(\sigma, \rho) & 0 \\ 0 & Q(\sigma, \rho) \end{pmatrix}.$$

The effective fields  $\psi, \theta, \chi$  and therefore, according to (5), the metric coefficients  $A, B, C$  are surely the same as in empty space [3]. Solutions of these equations have been obtained by the method of functional parameter (9). The fourth equation, describing  $\phi = \ln D$ , will have the form

$$\begin{aligned} \square \phi + \frac{1}{2}(\psi_z^2 - \psi_t^2) - \frac{1}{2}(\chi_z^2 - \chi_t^2) \sinh^2 \theta \\ - \frac{1}{2}(\theta_z^2 - \theta_t^2) + P(\sigma_z^2 - \sigma_t^2) + Q(\rho_z^2 - \rho_t^2) &= 0. \end{aligned} \quad (24)$$

Under the assumption (9) this equation may be written in the form

$$\ddot{\phi} + \frac{1}{2\xi^2}(a^2 - 1) + P(\sigma, \rho)\dot{\sigma}^2 + Q(\sigma, \rho)\dot{\rho}^2 = 0. \quad (25)$$

This equation may be analyzed either by direct integration by methods applicable to NSM or by the modified method of solutions generation from a vacuum seed solution [3] by putting  $\phi = \phi_{(v)} + \phi_{(m)}$ . Here  $\phi_{(v)}$  is a solution to the vacuum part of Eq. (24), corresponding to empty space (9). The term  $\phi_{(m)}$  is the matter correction and in the present case satisfies the equation

$$\square \phi_{(m)} + P(\sigma, \rho)(\sigma_z^2 - \sigma_t^2) + Q(\sigma, \rho)(\rho_z^2 - \rho_t^2) = 0. \quad (26)$$

The fields  $\sigma$  and  $\rho$  are solutions of (18) written for our NSM as follows:

$$\begin{aligned} P \square \sigma - \frac{1}{2} Q_\sigma (\rho_z^2 - \rho_t^2) + \frac{1}{2} P_\sigma (\sigma_z^2 - \sigma_t^2) \\ + P_\rho (\sigma_z \rho_z - \sigma_t \rho_t) + P(\psi_z \sigma_z - \psi_t \sigma_t) &= 0, \\ Q \square \sigma + \frac{1}{2} Q_\rho (\rho_z^2 - \rho_t^2) - \frac{1}{2} P_\rho (\sigma_z^2 - \sigma_t^2) \\ + Q_\sigma (\sigma_z \rho_z - \sigma_t \rho_t) + Q(\psi_z \rho_z - \psi_t \rho_t) &= 0. \end{aligned} \quad (27)$$

To solve these equations, let us apply the functional parameter method with  $\psi = \ln \xi$ ,  $\sigma = \sigma(\xi)$ ,  $\rho = \rho(\xi)$ . The set of ordinary differential equations set may be written as

$$\begin{aligned} \ddot{\sigma} - \frac{1}{2} \frac{Q_\sigma}{P} \dot{\rho}^2 + \frac{1}{2} (\ln P)_\sigma \dot{\sigma}^2 + (\ln P)_\rho \dot{\sigma} \dot{\rho} + \frac{\dot{\sigma}}{\xi} &= 0, \\ \ddot{\rho} - \frac{1}{2} \frac{P_\rho}{Q} \dot{\sigma}^2 + \frac{1}{2} (\ln Q)_\rho \dot{\rho}^2 + (\ln Q)_\sigma \dot{\sigma} \dot{\rho} + \frac{\dot{\rho}}{\xi} &= 0. \end{aligned} \quad (28)$$

These equations can hardly be exactly integrated for arbitrary  $P(\sigma, \rho)$  and  $Q(\sigma, \rho)$ . Here we discuss some special cases.

1. The simplest case,  $P(\sigma, \rho) = Q(\sigma, \rho) = 1$ :

$$\begin{aligned}\sigma(\xi) &= \sigma_0 + \sigma_1 \ln \xi, \\ \rho(\xi) &= \rho_0 + \rho_1 \ln \xi, \\ \phi &= \frac{1}{2}(a^2 - 1) + (\sigma_1^2 + \rho_1^2) \ln \xi + \phi_1 \xi + \phi_0.\end{aligned}\quad (29)$$

2.  $P(\sigma, \rho) = p(\sigma)$ ,  $Q(\sigma, \rho) = q(\sigma)$ :

$$\begin{aligned}\xi \frac{d\rho}{d\xi} &= \frac{c}{q(\sigma)}, \\ \left( \xi^2 \frac{d^2}{d\xi^2} + \xi \frac{d}{d\xi} \right) \sigma + \frac{1}{2} \xi^2 \frac{d \ln p(\sigma)}{d\xi} \frac{d\sigma}{d\xi} - \frac{c^2}{2pq^2} \frac{dq}{d\sigma} &= 0.\end{aligned}$$

2.1. Rotation surface,  $P(\sigma, \rho) = 1$ ,  $Q(\sigma, \rho) = q(\sigma)$ :

$$\xi \frac{d\rho}{d\xi} = \frac{c}{q(\sigma)}, \quad \frac{d\xi}{\xi} = \sqrt{k - \frac{c^2}{q(\sigma)}} d\sigma.$$

3.  $P(\sigma, \rho) = p(\sigma)$ ,  $Q(\sigma, \rho) = q(\rho)$ :

$$\sqrt{p(\sigma)} d\sigma = \sigma_0 \frac{d\xi}{\xi}, \quad \sqrt{q(\rho)} d\rho = \rho_0 \frac{d\xi}{\xi}. \quad (31)$$

This case is of particular interest. In the case (31) matter appears in the form

$$\ddot{\phi}_{(m)} = \frac{1}{\xi^2} (\sigma_0^2 + \rho_0^2),$$

which leads to

$$\phi_{(m)} = \phi_m^{(0)} + \phi_m^{(1)} \xi + (\sigma_0^2 + \rho_0^2) \ln \xi.$$

The above expression is similar to the vacuum case,

$$\phi_{(v)} = \phi_0 + \phi_1 \xi + \frac{a^2 - 1}{2} \ln \xi.$$

Then the solution is

$$\begin{aligned}\phi &= \left( \frac{a^2 - 1}{2} + \sigma_0^2 + \rho_0^2 \right) \ln \xi \\ &+ (\phi_m^{(1)} + \phi^{(1)}) \xi + (\phi_m^{(0)} + \phi^{(0)}).\end{aligned}\quad (32)$$

This expression for the field  $\phi$  coincides with the solution (30). This means that cases under consideration is a sort of gauge transformation of chiral space. Solution for  $\phi$  (for  $D$ ) in the presence of matter, by a proper choice of the constants (under special initial conditions) may lead to the same results as in the vacuum case. The chiral space corresponding to  $h_{AB}$  may be then represented as a non-deformed plane which may be curved in any way.

## 4 Conclusions

In this paper the effective NSM method has been extended to plane-symmetric space-times filled by chiral fields. The functional parameter method was shown to be applicable for generating exact solutions for such

systems. It opens possibilities of construction and study of exact cosmological solutions in the spirit of Belinskii [8], with sources represented by nonlinear sigma models. A family of such solutions was pointed out in [3], and any classes of solution may be constructed in the future by the method suggested.

## Acknowledgement

This work has been carried out under the aegis of the State Research Programme ‘‘Astronomy. Fundamental Space Research’’, Section ‘‘Cosmomicrophysics,’’ and with partial financial support from the Russian Basic Research Foundation (Grant No. 98-02-18040) and the Centre of Cosmoparticle Physics ‘‘Cosmion’’. The authors thank V.M. Zhuravlev for useful discussions.

## References

- [1] R.A. Matzner and C.W. Misner, *Phys. Rev.* **154**, 1229 (1967).
- [2] Chervon S.V., Muslimov A.G., *Phys.Lett.* **A142**, 1989, P.14.
- [3] S.V. Chervon, ‘‘Non-Linear Fields in the Theory of Gravitation and Cosmology’’, Middle-Volga Scientific Centre, Ulyanovsk State University, 1997.
- [4] S.V. Chervon, V.M. Zhuravlev and D.Yu. Shabalkin, *Grav. & Cosmol.* **3** (1997), No. 4(12), 312-316.
- [5] D.Yu. Shabalkin and S.V.Chervon, *Izvestiya Vuzov. Fizika* 1998, No 6 P.114 (in Russian)
- [6] Shabalkin D.Yu. *Grav. & Cosmol.* **4**, 4 (16), 302-307 (1998).
- [7] Shabalkin D.Yu., ‘‘Effective Nonlinear Sigma Models in Gravitation and Cosmology’’, Ulyanovsk, 1998, PhD thesis (in Russian).
- [8] V.A. Belinskii, *Zh. Eksp. Teor. Fiz.* **77**, 1239–1254 (1979).